ON THE EFFECT OF VISCOSITY AND THERMAL CONDUCTIVITY ON THE STRUCTURE OF COMPRESSIBLE FLOWS

(O VLIIANII VIAZKOSTI I TEPLOPROVODNOSTI NA STRUKTURU SZHIMAEMYKH TECHENII)

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An attempt to simplify Navier-Stokes equations for description of two-dimensional stationary nearsonic flows of a perfect (*) gas was apparently made for the first time by Liepman, Ashkanas and Cole (see for example [1]). The work of Sternberg [2], Sichel [3 and 4], Szaniawsky [5] and other authors has been devoted to a clarification of the effect of viscosity and thermal conductivity on some flows of a compressible medium with velocities close to the velocity of sound.

In the present paper asymptotic equations, which are satisfied by flows of the "short wave" type [6 and 7] propagating in a viscous thermally conducting medium are derived on the basis of Navier-Stokes equations and on the basis of fundamental laws of thermodynamics. Qualitative estimates are obtained for dimensions of zones in which the effect of dissipative processes can be substantial.

In the second part of the paper stationary near-sonic flows are examined. The effect of viscosity and thermal conductivity on asymptotic pattern of flow over profiles and bodies of revolution by a stream which is sonic at infinity is investigated. It is discovered that in the two-dimensional case the solution of Frankl' [8] for an ideal gas, correctly describes nearsonic flow of a real gas far away from the profile with the exception of the shock front structure itself. However, in the case of flow over bodies of revolution by a stream which is sonic at infinity, the asymptotic flow pattern of a viscous, thermally conducting gas is qualitatively different from the pattern given by the solution of equations for an ideal gas [9 and 10]. In order to establish the true flow pattern it is necessary to preserve dissipation terms in the equations of motion.

In the description of real gas flows it is frequently permissible to neglect viscosity and thermal conductivity because usually coefficients of viscosity and heat conductivity are not large. Dissipation processes play an important role only in regions where a sharp change in flow parameters occurs, for instance in the boundary layer. These same processes together with the

*) The term "perfect" is applied to a gas which obeys the Clapeyron equation of state, the designation "ideal" will refer to an inviscid, non-thermally-conducting gas.

nonlinear character of equations of gas dynamics determine the structure of shock waves [11 and 12]. Problems are frequently encountered in gas dynamics where fairly sharp changes in flow parameters take place over narrow regions adjacent to shock fronts. Such flows are referred to as short waves; their general theory for an ideal gas was developed in [6 and 7].

In short waves gradients of flow parameters can be so significant that it becomes necessary to take into account the effect of viscosity and thermal conductivity. In the paper of Sternberg [2] which is devoted to Mach reflections of weak shock waves, it is shown that discrepancies exist between theoretical and experimental results. This is related to the fact that the flow theory of an ideal fluid is not satisfactory in some region near the triple point and it is necessary to take into account dissipation processes which take place in the real gas. In this region splitting of the incident shock wave into a reflected wave and so-called Mach "stem" occurs. In the nontransition region each of the waves mentioned has quasi-one-dimensional structure. On the other hand the structure of shock waves in the vicinity of the triple point is essentially two-dimensional and is not permissible any more to neglect the change in the velocity vector component which is tangential to the front. Sternberg called such shock waves the "non-Hugoniot" type. It follows from conservation laws [2] that the width of the weak non-Hugoniot shock wave is at least several times greater than the width of the shock wave for which the usual conditions of the surface of a strong discontinuity apply. Another example of a non-Hugoniot shock wave arising in the interaction of a weak shock wave with a boundary layer was investigated by Sichel [3 and 4].

Problems investigated in [2 to 4] represent essentially examples of short waves arising in stationary flows where dissipation processes take place.

1. We will derive general equations for short waves in a viscous, thermally conducting gas. Equations of continuity, Navier-Stokes equations and energy equations in the case of two-dimensional nonstationary motion are written in the following form, respectively, [12 and 13]

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{(k-1)\,\rho v_y}{y} = 0$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left\{2\lambda\frac{\partial v_x}{\partial x} + \left(\zeta - \frac{2}{3}\lambda\right)\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{(k-1)\,v_y}{y}\right]\right\} + \frac{\partial}{\partial y}\left[\lambda\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{(k-1)\,\lambda}{y}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \quad (1.1)$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\lambda\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left\{2\lambda\frac{\partial v_y}{\partial y} + \left(\zeta - \frac{2}{3}\lambda\right)\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{(k-1)\,v_y}{y}\right]\right\} + \frac{2\,(k-1)\lambda}{y}\left(\frac{\partial v_y}{\partial y} - \frac{v_y}{y}\right)$$

$$\rho T\left(\frac{\partial s}{\partial t} + v_x\frac{\partial s}{\partial x} + v_y\frac{\partial s}{\partial y}\right) = \frac{\partial}{\partial x}\left(\varkappa\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varkappa\frac{\partial T}{\partial y}\right) + \frac{(k-1)\,\varkappa}{y}\frac{\partial T}{\partial y} + 2\lambda\left\{\left(\frac{\partial v_x}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left[\frac{(k-1)\,v_y}{y}\right]^2\right\} + \left(\zeta - \frac{2}{3}\lambda\right)\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{(k-1)\,v_y}{y}\right]^2 \quad (1.2)$$

Here t is time, x, y are orthogonal Cartesian (for plane-parallel flow) or cylindrical (for the case of symmetry with respect to x-axis) coordinates, v_x, v_y are corresponding components of the velocity vector, ρ is density, p is pressure, T is temperature, s is specific entropy, $\lambda, \zeta, \varkappa$

are coefficients of viscosity, second viscosity and thermal conductivity. For plane-parallel flow the coefficient $k = 1$, for axisymmetric flow $k = 2$. It is frequently convenient to examine the coefficient of so-called "longitudinal" viscosity $\mu = {}^4/_3\,\lambda + \zeta$.

We will introduce specific enthalpy w, adiabatic sound velocity a, thermal expansion coefficient α, thermal pressure coefficient β, specific heats at constant pressure c_p and constant volume c_v and the ratio of heat capacities γ

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s, \qquad \alpha = \rho\left(\frac{\partial}{\partial T}\frac{1}{\rho}\right)_p, \qquad \beta = \frac{1}{p}\left(\frac{\partial p}{\partial T}\right)_\rho, \qquad \gamma = \frac{c_p}{c_v}$$

According to equation of state and a basic thermodynamic relationship, the following equations are applicable for any two-parameter medium:

$$dp = a^2 d\rho + \left(\frac{\partial p}{\partial s}\right)_\rho ds, \qquad dw = T ds + \frac{dp}{\rho} \tag{1.3}$$

If one takes advantage of relationships between thermodynamic quantities [14]

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma \left(\frac{\partial p}{\partial \rho}\right)_T, \qquad c_p - c_v = \frac{T p \alpha \beta}{\rho} = \left[\frac{1}{\rho} - \left(\frac{\partial w}{\partial p}\right)_T\right]\left(\frac{\partial p}{\partial T}\right)_\rho$$

then Equation (1.3) can be rewritten in the form [15]

$$dp = a^2 d\rho + \frac{\alpha T \rho a^2}{c_p} ds, \qquad T ds = c_p dT - \frac{\alpha T}{\rho} dp, \qquad \alpha T = \frac{(\gamma - 1)\, c_p}{a^2 \alpha} \tag{1.4}$$

For a perfect gas it is well-known that $\alpha = 1/T$. Equations (1.1),(1.2) and (1.3) form a closed system.

Eliminating specific entropy s from Equations (1.2) and (1.4) we obtain

$$\rho c_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y}\right) - \alpha T \left(\frac{\partial p}{\partial t} + v_x \frac{\partial p}{\partial x} + v_y \frac{\partial p}{\partial y}\right) =$$
$$= \frac{\partial}{\partial x}\left(\varkappa \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varkappa \frac{\partial T}{\partial y}\right) + \frac{(k-1)\,\varkappa}{y}\frac{\partial T}{\partial y} + 2\lambda \left\{\left(\frac{\partial v_x}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2 + \right.$$
$$\left. + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left[\frac{(k-1)\, v_y}{y}\right]^2\right\} + \left(\zeta - \frac{2}{3}\lambda\right)\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{(k-1)\, v_y}{y}\right]^2 \tag{1.5}$$

$$\gamma dp = a^2 a\rho + \alpha \rho a^2 dT \tag{1.6}$$

It is assumed that a wave propagates in the direction of the x-axis through an undisturbed quiescent gas with parameters $p_0, \rho_0, T_0, a_0, \lambda_0, \zeta_0, \varkappa_0, \mu_0, c_{p_0}, \gamma_0$ and α_0. In this wave the excess values of all quantities are small compared to the initial values. Perturbations of pressure, density, temperature and speed of sound have the same order of smallness as the longitudinal component of the velocity vector v_x.

For derivation of approximate equations for short waves a moving system of coordinates is introduced just as in [6 and 7]

$$x = \xi t, \qquad y = \eta t, \qquad t = t \tag{1.7}$$

Transition is made to nondimensional variables

$$\xi = a_0(1 + \Delta_0\delta), \qquad \eta = a_0\theta_0\vartheta, \qquad t = \mu_0\tau_0\tau / \rho_0 a_0^2$$
$$v_x = a_0 M_0 u, \qquad v_y = a_0 N_0 v \tag{1.8}$$
$$p = \rho_0 a_0^2 (p_0 / \rho_0 a_0^2 + M_0 P), \qquad \rho = \rho_0(1 + M_0 R)$$
$$T = T_0(1 + M_0\Omega), \qquad a = a_0(1 + M_0 A)$$

Here M_0, N_0, Δ_0, θ_0 and τ_0 are characteristic values of velocity components, of coordinates and of time; $\delta, \vartheta, \tau, u, v, P, R, \Omega$ and A are quantities of the order of unity and Δ_0, M_0 and N_0 are small in comparison with unity.

It is known [12] that coefficients of viscosity and thermal conductivity usually have the same order of magnitude, i.e. their ratios are comparable in their order of magnitude to unity. Coefficients of viscosity and thermal conductivity are related to the value M_0. All other parameters are related to their values in the equilibrium state. Perturbations of all these values will be designated by primes

$$\lambda = \mu_0(\lambda_0 / \mu_0 + \lambda'), \ldots, \qquad \alpha = \alpha_0(1 + \alpha'), \ldots \tag{1.9}$$

Here $\lambda', \ldots, \alpha', \ldots$ are small compared to unity.

Passing to new variables (1.7) to (1.9) and preserving only major terms, the system of equations (1.1), (1.5) and (1.6) can be rewritten in the form

$$M_0\tau\frac{\partial R}{\partial\tau} + \frac{M_0}{\Delta_0}\left(\frac{\partial u}{\partial\delta} - \frac{\partial R}{\partial\delta}\right) + \frac{N_0}{\theta_0}\left[\frac{\partial v}{\partial\vartheta} + \frac{(k-1)v}{\vartheta}\right] = 0$$

$$M_0\tau\frac{\partial u}{\partial\tau} - \frac{M_0}{\Delta_0}\frac{\partial u}{\partial\delta} + \frac{N_0 M_0}{\theta_0}v\frac{\partial u}{\partial\vartheta} = -\frac{M_0}{\Delta_0}\frac{\partial P}{\partial\delta} + \frac{M_0}{\tau_0\Delta_0^2\tau}\frac{\partial^2 u}{\partial\delta^2} +$$
$$+ \frac{\lambda_0 M_0}{\tau_0\mu_0\theta_0^2\tau}\frac{\partial^2 u}{\partial\vartheta^2} + \frac{(\zeta_0 + {}^1/_3\lambda_0)N_0}{\tau_0\mu_0\Delta_0\theta_0\tau}\left[\frac{\partial^2 v}{\partial\delta\partial\vartheta} + \frac{k-1}{\vartheta}\frac{\partial v}{\partial\delta}\right] + \frac{(k-1)\lambda_0 M_0}{\tau_0\mu_0\theta_0^2\tau\vartheta}\frac{\partial u}{\partial\vartheta}$$
$$N_0\tau\frac{\partial v}{\partial\tau} - \frac{N_0}{\Delta_0}\frac{\partial v}{\partial\delta} + \frac{N_0^2}{\theta_0}v\frac{\partial v}{\partial\vartheta} = -\frac{M_0}{\theta_0}\frac{\partial P}{\partial\vartheta} + \frac{\lambda_0 N_0}{\tau_0\mu_0\Delta_0^2\tau}\frac{\partial^2 v}{\partial\delta^2} +$$
$$+ \frac{(\zeta_0 + {}^1/_3\lambda_0)M_0}{\tau_0\mu_0\Delta_0\theta_0\tau}\frac{\partial^2 u}{\partial\delta\partial\vartheta} + \frac{N_0}{\tau_0\theta_0^2\tau}\left[\frac{\partial^2 v}{\partial\vartheta^2} + \frac{k-1}{\vartheta}\left(\frac{\partial v}{\partial\vartheta} - \frac{v}{\vartheta}\right)\right]$$

$$\gamma_0\left(M_0\tau\frac{\partial P}{\partial\tau} - \frac{M_0}{\Delta_0}\frac{\partial P}{\partial\delta} + \frac{M_0 N_0}{\theta_0}v\frac{\partial P}{\partial\vartheta}\right) = M_0\tau\frac{\partial R}{\partial\tau} - \frac{M_0}{\Delta_0}\frac{\partial R}{\partial\delta} + \frac{M_0 N_0}{\theta_0}v\frac{\partial R}{\partial\vartheta} +$$
$$+ \alpha_0 T_0\left(M_0\tau\frac{\partial\Omega}{\partial\tau} - \frac{M_0}{\Delta_0}\frac{\partial\Omega}{\partial\delta} + \frac{M_0 N_0}{\theta_0}v\frac{\partial\Omega}{\partial\vartheta}\right) \tag{1.10}$$

$$\tau\left(\frac{\partial u}{\partial\tau}+\frac{\partial P}{\partial\tau}\right)+\left(\frac{M_0}{\Delta_0}u-\delta\right)\left(\frac{\partial u}{\partial\delta}+\frac{\partial P}{\partial\delta}\right)+\left(\frac{N_0}{\theta_0}v-\vartheta\right)\left(\frac{\partial u}{\partial\vartheta}+\frac{\partial P}{\partial\vartheta}\right)+ \qquad (1.10)\ \text{cont.}$$

$$+\frac{N_0}{M_0\theta_0}\left[\frac{\partial v}{\partial\vartheta}+\frac{(k-1)v}{\vartheta}\right]-\frac{1}{\tau_0\Delta_0{}^2\tau}\frac{\partial^2 u}{\partial\delta^2}-\frac{\lambda_0}{\tau_0\mu_0\theta_0{}^2\tau}\left[\frac{\partial^2 u}{\partial\vartheta^2}+\frac{k-1}{\vartheta}\frac{\partial u}{\partial\vartheta}\right]-$$

$$-\frac{(\zeta_0+{}^1/_3\,\lambda_0)N_0}{\tau_0\mu_0M_0\Delta_0\theta_0\tau}\left[\frac{\partial^2 v}{\partial\delta\partial\vartheta}+\frac{k-1}{\vartheta}\frac{\partial v}{\partial\delta}\right]-\frac{\gamma_0\gamma'}{\Delta_0}\frac{\partial P}{\partial\delta}+\frac{2M_0A}{\Delta_0}\frac{\partial R}{\partial\delta}+$$

$$+\frac{\alpha_0T_0\alpha'}{\Delta_0}\frac{\partial\Omega}{\partial\delta}+\frac{2\alpha_0T_0M_0A}{\Delta_0}\frac{\partial\Omega}{\partial\delta}-\frac{\alpha_0T_0c_p{}'}{\Delta_0}\frac{\partial\Omega}{\partial\delta}+\frac{M_0(\gamma_0-1)}{\Delta_0}\Omega\frac{\partial P}{\partial\delta}+$$

$$+\frac{(\gamma_0-1)\alpha'}{\Delta_0}\frac{\partial P}{\partial\delta}-\frac{\varkappa_0\alpha_0T_0}{\tau_0\mu_0\Delta_0{}^2c_{p_0}\tau}\frac{\partial^2\Omega}{\partial\delta^2}-\frac{\varkappa_0\alpha_0T_0}{\tau_0\mu_0\theta_0{}^2c_{p_0}\tau}\left[\frac{\partial^2\Omega}{\partial\vartheta^2}+\frac{k-1}{\vartheta}\frac{\partial\Omega}{\partial\vartheta}\right]-$$

$$-\frac{1}{2}\frac{N_0{}^2\lambda_0(\gamma_0-1)}{\tau_0\mu_0M_0\Delta_0{}^2\alpha_0T_0\tau}\left(\frac{\partial v}{\partial\delta}\right)^2-\frac{N_0{}^2(\gamma_0-1)}{\tau_0M_0\theta_0{}^2\alpha_0T_0\tau}\left\{\left(\frac{\partial v}{\partial\vartheta}\right)^2+\left[\frac{(k-1)v}{\vartheta}\right]^2\right\}-$$

$$-\frac{2N_0{}^2(\gamma_0-1)(\zeta_0-{}^2/_3\lambda_0)}{\tau_0\mu_0M_0\theta_0{}^2\alpha_0T_0\tau}\frac{(k-1)v}{\vartheta}\frac{\partial v}{\partial\vartheta}=0$$

The last equation of system (1.10) is the result of combining the first two equations of (1.1) with Equations (1.5) and (1.6) and substitution of Equations (1.7) to (1.9) into the relationship obtained. Such a procedure is connected with the fact that in weak waves the enthropy change which depends on terms on the left-hand side of Equations (1.2) and (1.5) has a higher order of smallness than the increases in other parameters. This means that in order to obtain a nontrivial equation for perturbations from the energy equation, it is necessary to exclude from it quantities of the first order of smallness which are connected with mass transport of substance and its impulse.

As had to be expected, due to substantial influence of viscosity and thermal conductivity no one problem can be of the similarity type because the time τ enters into coefficients of all dissipation terms of Equations (1.10), and derivatives with respect to time in these equations cannot become zero simultaneously.

It is characteristic of short waves that the velocity component and derivatives of all flow parameters in the direction of wave motion exceed in magnitude the velocity component and corresponding derivatives in the transverse direction. Thus it is possible to assume [6 and 7]

$$N_0 \ll M_0, \qquad \Delta_0 \ll \theta_0 \qquad (1.11)$$

Just as in ideal gas we will presume that the following expressions apply

$$\tau\partial R\,/\,\partial\tau, \qquad \tau\partial u\,/\,\partial\tau, \ldots \sim 1 \qquad (1.12)$$

In the first four equations of the system (1.10) therefore terms containing derivatives with respect to time are small compared to principal terms and they can be neglected. We note that retaining of these derivatives corresponds to the usual linearization of equations of motion of a viscous thermally conducting gas.

The characteristic value of time τ_0 was determined from Equations (1.8) as a quantity inversely proportional to the coefficient of longitudinal viscosity μ_0, which is usually proportional to the length of free path for molecules [12]. The wave length Δ_0 usually substantially exceeds the length of molecular free path. These considerations lead to inequality

$$1 / \tau_0 \ll \Delta_0 \tag{1.13}$$

Relationships (1.11) to (1.13) permit significant simplification of the system of equations (1.10) which after discarding of minor terms and after integration of the first, second and fourth equations takes the form

$$R = u, \qquad P = u, \qquad \Omega = \frac{\gamma_0 - 1}{\alpha_0 T_0} u, \qquad \frac{N_0}{\Delta_0}\frac{\partial v}{\partial \delta} = \frac{M_0}{\theta_0}\frac{\partial u}{\partial \vartheta}$$

$$\tau \frac{\partial u}{\partial \tau} + \left[\frac{M_0}{\Delta_0}(u + A) - \delta\right]\frac{\partial u}{\partial \delta} - \vartheta \frac{\partial u}{\partial \vartheta} + \frac{1}{2}\frac{N_0}{M_0\theta_0}\left[\frac{\partial v}{\partial \vartheta} + \frac{(k-1)\,v}{\vartheta}\right] -$$

$$- \frac{1}{2\tau_0 \Delta_0{}^2 \tau}\left[1 + \frac{(\gamma_0 - 1)\,\varkappa_0}{\mu_0 c_{p_0}}\right]\frac{\partial^2 u}{\partial \delta^2} = 0 \tag{1.14}$$

In the integration it was additionally assumed that the wave propagates through a homogeneous medium at rest. In the last equation the dependence which arises from the last condition (1.4), between perturbed values of gas parameters is also taken into account. The first three relationships (1.14) are analogous to relations between excess values of gas parameters in an acoustic wave or in a two-dimensional travelling impulse of small amplitude.

For simplicity we further suppose that $\Delta_0 \sim M_0$. Then it is possible to obtain from the last two equations of system (1.14)

$$\Delta_0 \sim M_0, \qquad \frac{N_0 \theta_0}{M_0{}^2} \sim 1, \qquad \tau_0 M_0{}^2 \sim 1, \qquad \frac{N_0}{M_0 \theta_0} \lesssim 1 \tag{1.15}$$

If $\theta_0 \sim 1$, then $N_0 \sim M_0{}^2$, and $N_0 / M_0 \theta_0 \sim M_0$. This case which belongs to quasi-one-dimensional flows, is not examined here. If however $\theta_0 \ll 1$, then it follows from relationships (1.15)

$$\theta_0 \sim \sqrt{M_0}, \qquad N_0 \sim M_0 \sqrt{M_0}, \qquad \tau_0 \sim 1 / M_0{}^2 \tag{1.16}$$

The first two conditions (1.16) are analogous to relationships between characteristic values of corresponding quantities in the theory of short waves for an ideal gas [6 and 7]. In the approximation investigated

$$A + u = m_0 u, \qquad m = \frac{1}{2\rho^3 a^2}\left(\frac{\partial^2 p}{\partial (1/\rho)^2}\right)_s$$

For a perfect gas $m = \frac{1}{2}(\gamma + 1)$. Taking into consideration the last relationship and substituting Equations (1.15) into the last two equations of (1.14) we arrive at equations of short waves propagating in a viscous thermally conducting gas

$$\frac{\partial u}{\partial \vartheta} = \frac{\partial v}{\partial \delta}, \qquad \tau \frac{\partial u}{\partial \tau} + (m_0 u - \delta)\frac{\partial u}{\partial \delta} - \vartheta \frac{\partial u}{\partial \vartheta} + \frac{1}{2}\left[\frac{\partial v}{\partial \vartheta} + \frac{(k-1)\,v}{\vartheta}\right] -$$

$$- \frac{1}{2\tau}\left(1 + \frac{\gamma_0 - 1}{N_{\mathrm{Pr}_0}}\right)\frac{\partial^2 u}{\partial \delta^2} = 0 \qquad \left(N_{\mathrm{Pr}} = \frac{\mu c_p}{\varkappa} = \frac{(4/3\lambda + \zeta)\, c_p}{\varkappa}\right) \tag{1.17}$$

Here N_{Pr} is the Prandtl number. The first of Equations (1.17), just as in the ideal gas, expresses the condition of irrotational flow. In the second equation, however, an additional last term appears which takes into account the viscosity and thermal conductivity of a real gas.

According to (1.17) the potential $\varphi(\delta, \vartheta, \tau)$ may be introduced for components of velocity u and v This potential must satisfy Equation

$$\tau \frac{\partial^2\varphi}{\partial\tau\partial\delta} + \left(m_0 \frac{\partial\varphi}{\partial\delta} - \delta\right) \frac{\partial^2\varphi}{\partial\delta^2} - \vartheta \frac{\partial^2\varphi}{\partial\delta\partial\vartheta} + \frac{1}{2}\left[\frac{\partial^2\varphi}{\partial\vartheta^2} + \frac{k-1}{\vartheta}\frac{\partial\varphi}{\partial\vartheta}\right] -$$

$$- \frac{1}{2\tau}\left(1 + \frac{\gamma_0 - 1}{N_{\mathrm{Pr}_0}}\right)\frac{\partial^3\varphi}{\partial\delta^3} = 0 \qquad \left(u = \frac{\partial\varphi}{\partial\delta},\ v = \frac{\partial\varphi}{\partial\vartheta}\right) \qquad (1.18)$$

With variables

$$\delta^\circ = \delta, \qquad \vartheta^\circ = \sqrt{2}\vartheta, \qquad \tau^\circ = 2\,[1 + (\gamma_0 - 1) / N_{\mathrm{Pr}_0}\,]^{-1}\,\tau$$

$$u^\circ = m_0 u, \qquad v^\circ = (m_0 / \sqrt{2})\, v, \qquad \varphi^\circ = m_0\varphi$$

Equations (1.17) and (1.18) take, correspondingly, the form (the superscripts $^\circ$ for variables are omitted in the following)

$$\frac{\partial u}{\partial\vartheta} = \frac{\partial v}{\partial\delta}, \qquad \tau\frac{\partial u}{\partial\tau} + (u - \delta)\frac{\partial u}{\partial\delta} - \vartheta\frac{\partial u}{\partial\vartheta} + \frac{\partial v}{\partial\vartheta} + \frac{(k-1)\,v}{\vartheta} - \frac{1}{\tau}\frac{\partial^2 u}{\partial\delta^2} = 0 \qquad (1.19)$$

$$\tau\frac{\partial^2\varphi}{\partial\tau\partial\delta} + \left(\frac{\partial\varphi}{\partial\delta} - \delta\right)\frac{\partial^2\varphi}{\partial\delta^2} - \vartheta\frac{\partial^2\varphi}{\partial\delta\partial\vartheta} + \frac{\partial^2\varphi}{\partial\vartheta^2} + \frac{k-1}{\vartheta}\frac{\partial\varphi}{\partial\vartheta} - \frac{1}{\tau}\frac{\partial^3\varphi}{\partial\delta^3} = 0 \qquad (1.20)$$

By means of equations obtained we can perform an asymptotic evaluation of size of the region in which dissipation processes play a substantial role.

For $\tau \to \infty$ all principal terms of Equation (1.19) must be comparable in the order of magnitude.

From this we have for $\delta^\times$ and $\vartheta^\times$ and for the duration $\tau^\times$ of the non-Hugoniot wave the relations

$$\delta^\times\tau \sim \eta^\times\tau \sim \tau^\times \sim \mathrm{const}$$

or

$$\delta^\times \sim \eta^\times \sim \mathrm{const} / \tau, \qquad \tau^\times \sim \mathrm{const} \qquad (1.21)$$

In accordance with Equations (1.7) and (1.8)

$$\delta^\times = x^\times / \tau, \qquad \vartheta^\times = y^\times / \tau \qquad\qquad (x^\times = (x - a_0 t)/a_0)$$

Here $x^\times$ and $y^\times$ represent the dimensions of the non-Hugoniot region in the physical space x and y

Conditions (1.21) now lead to important asymptotic evaluations

$$x^\times \sim y^\times \sim \tau^\times \sim \mathrm{const} \qquad (1.22)$$

This indicates that the dimensions and duration of non-Hugoniot wave tend to approach constant values for large values of time.

In application to the problem of Mach reflection of a weak shock wave from a wedge or conical point Equations (1.19) describe the transition process from the moment of formation of triple configuration to the point where the flow reaches some quasi-stationary state in the non-Hugoniot wave which has asymptotically constant dimensions and duration. For $\tau \to \infty$ certain similarity solutions of equations for short waves in an ideal gas [6 and 7] describe the entire flow region with the exception of the immediate vicinity of the triple point. At the triple point itself in these equations the presence of a singularity should be allowed.

In gas flow with velocities close to sound velocity the change of flow parameters is sufficiently gradual far away from bodies over which the flow occurs. But, just as in the case of short waves, the vector component of the perturbed velocity and derivatives of all parameters of the medium in the direction of free stream exceed significantly in magnitude the velocity component and the corresponding derivatives in the perpendicular direction. In other words, conditions analogous to (1.11) [9] are applicable. Here, as will be shown below, at sufficiently small Reynolds numbers and Peclet numbers the dissipation processes in the real gas can also have a substantial influence on the entire pattern of flow.

2. We will examine the problem of flow over a body by a viscous thermally conducting gas stream which is sonic at infinity. Derivation of equations for two-dimensional nonstationary nearsonic flow is analogous to derivation of equations for short waves presented in Section 1. The same set of Equations (1.1), (1.5) and (1.6) is selected as a starting point, only flow parameters are now related not to initial but to critical values which will be designated by an asterisk in the following text.

Nondimensional coordinates of time

$$x = \frac{\mu_* \Delta_0}{\rho_* a_*} x^\circ, \qquad y = \frac{\mu_* \theta_0}{\rho_* a_*} y^\circ, \qquad t = \frac{\mu_* t_0}{\rho_* a_*^2} t^\circ \tag{2.1}$$

and also components of velocity vector of the perturbed flow

$$v_x = a_* (1 + M_0 u), \qquad v_y = a_* N_0 v \tag{2.2}$$

are introduced somewhat differently.

Here again the nondimensional parameters x°, y°, t°, u and v are in the order of magnitude comparable to unity, M_0 and N_0 are small, the quantities Δ_0, θ_0, t_0 may be large. All other nondimensional parameters and their perturbations are introduced in analogy to Equations (1.8) and (1.9).

Substituting new variables into Equations (1.1), (1.5) and (1.6), taking into account conditions (1.11) and retaining major terms in relationships obtained, we have in analogy to Section 1

$$R = -u, \qquad P = -u, \qquad \Omega = -\frac{(\gamma_* - 1)}{\alpha_* T_*} u, \qquad \frac{N_0}{\Delta_0}\frac{\partial v}{\partial x} = \frac{M_0}{\theta_0}\frac{\partial u}{\partial y}$$

$$\frac{M_0}{t_0}\frac{\partial u}{\partial t} + \frac{M_0^2}{\Delta_0} m_* u \frac{\partial u}{\partial x} - \frac{N_0}{2\theta_0}\left[\frac{\partial v}{\partial y} + \frac{(k-1)\, v}{y}\right] - \frac{M_0}{2\Delta_0^2}\left(1 + \frac{\gamma_* - 1}{N_{\mathrm{Pr}_*}}\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{2.3}$$

Here the superscripts $^\circ$ are omitted for nondimensional variables.

The first three equations are integrated under the assumption of homogeneity of free stream. The last equation is a consequence of the first two equations (1.1) and Equations (1.5) and (1.6). Just as in Section 1 it is assumed that the characteristic time t_0 is sufficiently large, so that in all equations (2.3) with the exception of the last it is possible to neglect derivatives with respect to time. Otherwise a linearized system of acoustical equations with consideration of viscosity and thermal conductivity would have resulted.

Nonlinear equations of nonstationary nearsonic flows of a viscous thermally conducting gas are obtained from system (2.3) for conditions

$$\Delta_0 \sim 1 / M_0, \quad \theta_0 \sim 1 / M_0^{3/2}, \quad N_0 \sim M_0^{3/2}, \quad t_0 \sim 1 / M_0^2 \quad (2.4)$$

With new variables

$$x' = x, \quad y' = [1 + (\gamma_* - 1) / N_{\mathrm{Pr}_*}]^{1/2} y, \quad t' = {}^1/_2 [1 + (\gamma_* - 1) / N_{\mathrm{Pr}_*}]\, t$$

$$u' = 2m_* [1 + (\gamma_* - 1) / N_{\mathrm{Pr}_*}]^{-1} u, \quad v' = 2m_* [1 + (\gamma_* - 1) / N_{\mathrm{Pr}_*}]^{-3/2} v$$

they have the form (primes are omitted everywhere in the following text)

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{(k-1)\, v}{y} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (2.5)$$

As in Section 1 we can show that the entropy change is a quantity of higher order of smallness than perturbations of other flow parameters.

Nearsonic flows subject to Equations (2.5) are irrotational. For the potential of perturbed flow $\varphi(x,y,t)$ the following equation is valid

$$\frac{\partial^2 \varphi}{\partial t \partial x} + \frac{\partial \varphi}{\partial x} \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} - \frac{k-1}{y} \frac{\partial \varphi}{\partial y} - \frac{\partial^3 \varphi}{\partial x^3} = 0 \quad \left(u = \frac{\partial \varphi}{\partial x}, \; v = \frac{\partial \varphi}{\partial y}\right) \quad (2.6)$$

In the papers of Frankl' [8] and Landau and Lifshits [12] the stationary problem of flow over profiles by a stream of ideal gas which is sonic at infinity, is solved. The analogous problem for flow over bodies of revolution is examined by Guderley, Yoshihara and Barish [9, 16 and 17], and by Fal'kovich and Chernov [10]. In both cases the flow far from the body is described by a similarity solution

$$\varphi(x, y) = y^{3n-2}\, \Phi(\xi) \qquad (\xi = x/y^n) \quad (2.7)$$

where for flow over a finite body the value $n = {}^4/_5$ corresponds to profiles and $n = {}^4/_7$ corresponds to bodies of revolution.

For smaller values of n in the solution (2.7) in both cases a limiting line arises. If the similarity index varies in the range ${}^4/_5 < n < 1$ for two-dimensional flow and ${}^4/_7 < n < 1$ for axisymmetric flow, then the problem of flow over a half-body which is expanding in width to infinity is obtained.

An example of nearsonic flow of an ideal gas near a half-body of revolution of the form $Y \sim \sqrt{x}$ is given by Ladyzhenskii [18], who found the exact solution of an equation for the potential $\Phi(\xi)$ in the case $n = {}^2/_3$.

The flow of real gases can be described by solutions for equations of motion of an ideal compressible fluid only in the case when they turn out to be asymptotical for equations of motion of real gases (1.1), (1.5) and (1.6) for small values of coefficients of viscosity and thermal conductivity. From this point of view it is interesting to examine the similarity solution (2.7).

If one substitutes Expression (2.7) into Equation (2.6) which for the stationary case has the form

$$\frac{\partial \varphi}{\partial x} \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} - \frac{k-1}{y} \frac{\partial \varphi}{\partial y} - \frac{\partial^3 \varphi}{\partial x^3} = 0 \quad (2.8)$$

and lets y go to infinity for finite values of x, then terms of the Equation (2.8) will de rease as various powers of y

$$\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}, \quad \frac{\partial^2\varphi}{\partial y^2}, \ldots \sim y^{3n-4}, \quad \frac{\partial^3\varphi}{\partial x^3} \sim y^{-2} \tag{2.9}$$

From this it is evident that for $n > {}^2/_3$ the dissipation term in (2.8) is small compared to other terms and equations of motion for an ideal gas can be used. If however $n \leqslant {}^2/_3$, then the last term in the equation (2.8) becomes comparable in magnitude to the first term or even larger. Substitution of Expression (2.7) into initial equations (1.1), (1.5) and (1.6) leads to the same result.

For the flow of (2.7) we can examine the Reynolds number N_{Re} and the Peclet number N_{Pe}, which characterize the effect of viscosity and thermal conductivity

$$N_{\mathrm{Re}} = \rho_* UX / \mu_*, \qquad N_{\mathrm{Pe}} = \rho_* UX / \varkappa_* \tag{2.10}$$

Here U and X are the characteristic values of velocity and length. In the stationary problem (2.6) and (2.7) concerning asymptotic rules of decay of perturbations far from bodies over which the flow takes place, the characteristic velocity is the velocity of perturbed motion. From relationships (2.6) and (2.7) for components of velocity U and V far from the body the following result is obtained

$$U \sim y^{2(n-1)}, \qquad V \sim y^{3(n-1)} \tag{2.11}$$

and the characteristic length $X \sim y^n$.

Taking into account these relationships the following asymptotic estimates are applicable

$$N_{\mathrm{Re}}, N_{\mathrm{Pe}} \sim y^{3n-2} \tag{2.12}$$

For $n > {}^2/_3$ Reynolds number and Peclet number are large and dissipation processes may be neglected, however for $n \leqslant {}^2/_3$ they are finite or even small. This indicates that even for fairly gradual change in flow parameters it is necessary to take into consideration viscosity and thermal conductivity.

Thus the solution of the two-dimensional problem by Fankl' [8] for near-sonic flow of ideal gas over finite bodies ($n = {}^4/_5$) can be utilized in the case of a real gas with the exception of description of the shock front structure itself which was introduced into this solution by Landau and Lifshits [12]. The asymptotic pattern of flow over finite bodies of revolution ($n = {}^4/_7$) by a gas stream which is sonic at infinity is qualitatively different from the pattern which is obtained in the framework of equations of motion of an ideal gas, and it must be established on the basis of the solution of the full Equation (2.8).

It is also important to emphasize that, as is evident from relationships (2.11), consideration of viscosity and thermal conductivity in flow over finite bodies of revolution leads to a lowering of the degree of decay of velocity components with distance.

A change in the asymptotic pattern of flow and a decrease in the degree of decay of perturbations as compared to an ideal gas will also take place in half-bodies of revolution corresponding to values in the interval $^4/_7 < n < ^2/_3$.

For stationary flows Equations (2.5) assume the form

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \qquad u\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{(k-1)\,v}{y} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.13}$$

The system of equations (2.13) permits a similarity solution of the type (2.7) for a single value $n = ^2/_3$. In order to examine this case we introduce new variables

$$u\,(x,\,y) = y^{-2/3} f\,(\xi), \qquad v\,(x,\,y) = y^{-1} g\,(\xi) \qquad (\xi = x\,/\,y^{2/3}) \tag{2.14}$$

Functions $f(\xi)$ and $g(\xi)$ must satisfy the equations

$$\frac{d^2 f}{d\xi^2} - f\frac{df}{d\xi} - \frac{2}{3}\xi\frac{dg}{d\xi} + (k-2)\,g = 0, \qquad \frac{dg}{d\xi} = -\frac{2}{3}f - \frac{2}{3}\xi\frac{df}{d\xi} \tag{2.15}$$

Integrating the last equation and substituting the obtained relationship into the the first equation, the system (2.15) can be represented in the form

$$\frac{d^2 f}{d\xi^2} + \left(\frac{4}{9}\xi^2 - f\right)\frac{df}{d\xi} + \frac{4}{9}\xi f + \frac{2}{3}(k-2)(c - f\xi) = 0, \qquad g = \frac{2}{3}(c - f\xi) \tag{2.16}$$

For the solution of the ordinary differential equation (2.16) in the axisymmetric case ($k = 2$) numerical methods were applied. Results of integration were compared with solution [18] of the problem of flow over half-bodies of revolution by a stream of ideal gas which is sonic at infinity. Boundary conditions were selected in such a manner that in the solution of Equation (2.16) the outer limit of the boundary layer coincides with the walls of the half-body $Y = Y(x)$ in the ideal gas. The relationship $Y(x)$ is determined by the index of similarity $n = ^2/_3$ and by the value of constant c in (2.16)

$$Y = \sqrt{^8/_3 c x}$$

In Fig.1 and 2 in the upper and lower half-plane, respectively, plots of dependence of functions f and g on the similarity variable ξ are presented. Lines marked with index 1 correspond to solutions of Equations (2.16), with index 2 correspond to results of [18]. In flow around "thick" half-bodies which correspond to relatively large values of c ($c = 1.2$; Fig.1) the differences between real and ideal gases are not great. It may only be noted that the peak of the function f is slightly cut off and shifted in the direction of flow. In the profile of the function g , on the other hand, arises a so far weakly developed maximum. With decreasing relative "thickness" of the half-body ($c = 0.0149$; Fig.2) these differences become substantial and dissipation processes play an increasing role. The increase in the effect of viscosity and thermal conductivity in the flow over thinner half-bodies is related to the circumstance that for decrease in the value of c , the values of Reynolds number and Peclet number (2.10) also decrease. In fact, it follows from group properties of solution (2.14) for an ideal gas that the functions

$$u = K^2 y^{-2/3} f(K\xi), \qquad v = K^3 y^{-1} g(K\xi) \tag{2.17}$$

where K is an arbitrary constant, also represent solutions of equations of nearsonic motion of an ideal gas. Boundary conditions permit to establish

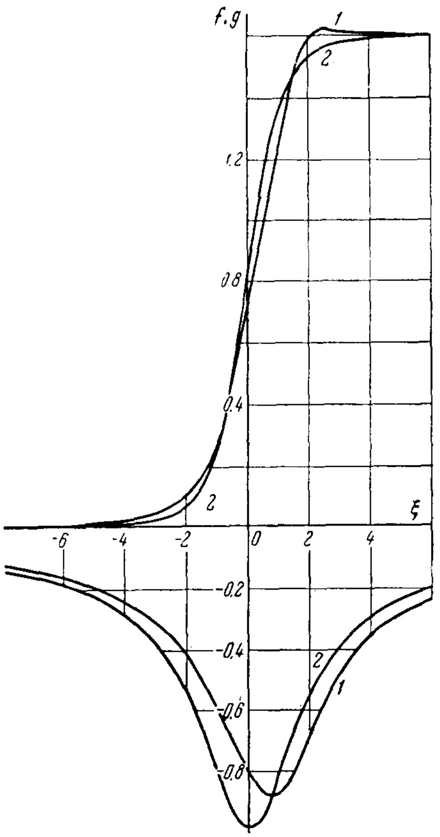

Fig. 1

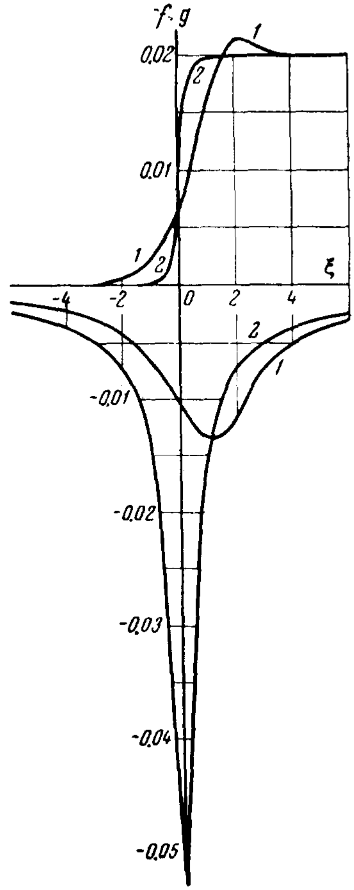

Fig. 2

a connection between constants K and c for the half-bodies under examination

$$K = (4/_3 c)^{1/3}$$

Now it is possible to find the dependence between Reynolds and Peclet numbers and the quantity c

$$N_{\mathrm{Re}}, N_{\mathrm{Pe}} \sim K^2 y^{-2/3} K^{-1} y^{2/3} \sim K \sim c^{1/3} \tag{2.18}$$

From relationship (2.18) it is easy to see that the thinner the half-body (i.e. the smaller the quantity c), the smaller will be the asymptotic values of N_{Re} and N_{Pe} numbers and the more substantial will be the influence of viscosity and thermal conductivity. As was to be expected, after estimates (2.9), for flow over half-bodies of the form $Y \sim \sqrt{x}$ (which corresponds to a similarity index $n = {}^2/_3$) corrections of solution [18] on account of viscosity and thermal conductivity have the same order of magnitude as the solution itself.

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